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Shannon-Type Sampling Theory on Unions of Equally Spaced and Noncommensurate Grids

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Abstract

Sampling theory is that branch of mathematics that seeks to reconstruct functions from its values at a discrete set of points. The fundamental result in sampling theory known as "Shannon's sampling theorem" has many applications to signal processing and communications engineering. I demonstrate Shannon's result via complex interpolation methods. I then quote a result that uses these methods to solve interpolation problems on unions of noncommensurate lattices, which are created via a specific number of theoretic guidelines. These interpolations give Shannon-type reconstructions on these lattices. I close by doing simulations in MATLAB of the sampling reconstructions on these noncommensurate grids.

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1. Introduction

Sampling theory is the distinctive branch of mathematics that attempts to solve the interpolation problem of a function with fixed growth from known sampled values by reconstructing the original function. The theory is a subset of the general theory of interpolation, which constructs functions that satisfy the known values. Sampling theory uses interpolation with other knowledge of a function to find methods that reconstruct the original function. This involves restrictions on the samples of known values.

The most significant contribution to this theory is called the "classical sampling theorem." This theorem is known by a myriad of names—Whittaker, Kotel'nikov, and Shannon—but the result is essentially the same. The theorem tells us that if a signal f is a function of finite energy on $\mathbb{R}$ ($f \in L^2(\mathbb{R})$) with Fourier transform $\widehat{f}(\omega) = 0$ for all $|\omega| \geq \Omega$, and if $T \leq 1/2\Omega$, then for all $t \in \mathbb{R}$,

$$f(t) = T \sum_{n=-\infty}^{\infty} f(nT) \frac{\sin(\frac{\pi}{T}(t - nT))}{\pi(t - nT)} \quad . \tag{1}$$

In other words, if f is sampled uniformly at a sufficiently high rate, f can be reconstructed from these samples by the formula in equation (1). This formula has a rich history extending back as far as the 1600s. However, much of the crucial research in developing and establishing the theory has occurred in the twentieth century, primarily because of the research in the various engineering fields. With the motivation of obtaining useful and practical results, the classical sampling theorem has been generalized and extended to a number of applications.

2. Preliminaries in Harmonic Analysis

Before I discuss the main results of this report, I first introduce a number of preliminary definitions and theorems. Much of what follows is standard notation in the literature and can be found in any text covering Fourier analysis. (For example, see Benedetto [1], Dym and McKean [2], or Körner [3]. Also see Apostol [4], Higgins [5], Marks [6], and Zayed [7].) First, I define in the following what are absolutely and square integrable functions:

Function f. A function f is called absolutely integrable, i.e., $f \in L^1(\mathbb{R})$, if

$$\int_{\mathbb{R}} |f(x)|\, dx \equiv ||f||_1 < +\infty \ .$$

If f is in L^1, one can say that $||f||_1$ is the L^1 norm of f. Similarly, a function is called square integrable, i.e., $f \in L^2(\mathbb{R})$, if

$$\int_{\mathbb{R}} |f(x)|^2\, dx \equiv ||f||_2 < +\infty \ .$$

If f is in L^2, one can say that $||f||_2$ is the L^2 norm of f.

In this section, all functions are considered absolutely and square integrable functions on the real line, unless otherwise noted. Likewise, all integrals are assumed to be over the whole domain (either $\mathbb{R}$ or $\mathbb{C}$, depending on the context) unless noted otherwise. The Fourier series and Fourier transform play integral parts in the sampling theory and Fourier analysis. Their definitions, from Benedetto [1] and Dym and McKean [2], are defined in the following:

Fourier Series. Let f be a periodic, integrable function on $\mathbb{R}$, with period 2Ω. Then the Fourier series of f is represented as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp^{-i\pi nx/\Omega} \ , \tag{2}$$

where the Fourier coefficients c_n are defined by

$$c_n = \frac{1}{2\Omega} \int_{-\Omega}^{\Omega} f(x) \exp^{i\pi nx/\Omega}\, dx \ . \tag{3}$$

Fourier Transform and Inversion Formula. The Fourier transform of $f \in L^1(\mathbb{R})$ is defined as

$$\widehat{f}(\omega) = \int_{\mathbb{R}} f(t) \exp^{-2\pi i t\omega}\, dt \ , \tag{4}$$

and its inversion formula for $\widehat{f} \in L^1(\widehat{\mathbb{R}})$ is

$$f(t) = \int_{\mathbb{R}} \widehat{f}(\omega) \exp^{2\pi i \omega t} d\omega \ . \tag{5}$$

The factor 2π in the exponential simplifies Plancheral's formula, i.e., $||f||_2 = ||\widehat{f}||_2$. Often, the Fourier transform of f is also denoted F. The Fourier transform can be extended to square integrable functions via a density argument on $C^{\infty}_{\searrow}(\mathbb{R})$, the space of infinitely differentiable, rapidly decreasing functions on $\mathbb{R}$. The transform is an isometry in $L^2(\mathbb{R})$. By itself, the Fourier transform has useful applications in signal and image processing. If I consider the function f to be a signal, then I refer to the domain of f as the time domain and the domain of the transform as representing the signal frequency. The following definitions are useful when referring to the time or frequency domain of a signal or function:

Support. The support of a function $f : \mathbb{R} \to \mathbb{R}$, denoted supp($f$), is the closure of the set on which f is nonzero, i.e., supp(f) $= \overline{\{x \in \mathbb{R} : f(x) \neq 0\}}$. The function f is said to have compact support if supp(f) is a compact set.

Band-limited. The function $f : \widehat{\mathbb{R}} \to \widehat{\mathbb{R}}$ is called Ω band-limited if the support of its Fourier transform $\widehat{f}$ is contained in the bounded interval $[-\Omega, \Omega]$ in $\widehat{\mathbb{R}}$, i.e., supp($\widehat{f}$) $\subset [-\Omega, \Omega]$.

The Fourier transform has a multitude of mathematically convenient and useful algebraic properties: linearity, symmetry, conjugation, translation (time shifting), modulation (frequency shifting), and time dilation [1]. Also, the transform has a number of important analytic properties: boundedness, continuity, Riemann-Lebesgue Lemma, time differentiation, and frequency differentiation [1–3,5]. One of the more relevant properties of the Fourier transform involves a mathematical operation called a convolution.

Convolution. For $f, g \in L^1(\mathbb{R})$, the convolution of f and g is defined by

$$h(x) = f * g(x) = \int f(y) g(x-y) \ dy \tag{6}$$

for $x \in \mathbb{R}$.

Convolution is a mathematical operation that consists of the filtering of one signal (the source) passing it through a filter, which is simply another signal.

Convolution is commutative, i.e., $f * g = g * f$, and if f, g, and h are of compact support, convolution is associative. The subject of convolution is important to signal reconstruction. The operation is the filtering of one function by passing it through a filter, which is simply another function. More on this topic will be discussed in section 4.1. For now, I note the following two significant properties from the Fourier perspective:

$$\begin{aligned} \widehat{f * g} &= \widehat{f} \cdot \widehat{g}, \text{ and} \\ \widehat{f \cdot g} &= \widehat{f} * \widehat{g} \, . \end{aligned}$$

These properties prove to be extremely useful in developing the theory on deconvolution.

Finally, I conclude with the Paley-Wiener theorem:

Paley-Wiener. The Fourier-Laplace transform of an infinitely differentiable function f with compact support contained in $\{|t| \leq A\}$ is an entire function $\widehat{f}(\zeta)$ in $\mathbb{C}$, which satisfies the following property:

For every integer $N \geq 0$, a positive constant $C = C(N)$ exists, such that

$$|\widehat{f}(\zeta)| \leq C(1 + |\zeta|)^{-N} \exp^{2\pi A|\mathrm{Im}\zeta|} \text{ for all } \zeta \in \mathbb{C} \ . \tag{7}$$

Conversely, every entire function in $\mathbb{C}$ satisfying this property is the Fourier-Laplace transform of a C^∞ function, with compact support within $\{|t| \leq A\}$.

The classical Paley-Wiener theorem says that a square-integrable complex-valued function, defined over the real line, can be extended off the real line as an entire function of exponential type if and only if its Fourier transform $f(x)$ is identically zero for $|x| > a$, i.e., if and only if f is band-limited to $[-a, a]$. (For a derivation, see literature by Dym and McKean [2].)

The Shannon series is a means of extending to the entire complex plane $\mathbb{C}$. The extension of the Paley-Wiener theorem to generalized functions (to tempered distributions) is called the Paley-Wiener-Schwartz theorem. Additional background material can be found in literature by Benedetto [1], Dym and McKean [2], and Körner [3].

3. Classical Sampling Theory

3.1 Classical Sampling Theorem

Sampling theory is an area of high interest and interacts with numerous other mathematical fields. The idea behind sampling is to convert continuous time signals into appropriate discrete signals that represent the original signal and vice versa. Reconstruction reverses the process to retrieve the original signal from the samples. This theory of sampling was created via an amalgam of mathematical, scientific, and engineering research. Without dispute, the fundamental result from the conglomeration of these sources has been what is called the classical sampling theorem: Let f be a function of finite energy on $\mathbb{R}$ ($f \in L^2(\mathbb{R})$) with Fourier transform $\widehat{f}(\omega) = 0$ for all $|\omega| \geq \Omega$, i.e., $f(t)$ is Ω band-limited.

If $T \leq 1/2\Omega$, then for all $t \in \mathbb{R}$,

$$f(t) = T \sum_{n=-\infty}^{\infty} f(nT) \frac{\sin(\frac{\pi}{T}(t - nT))}{\pi(t - nT)} . \tag{8}$$

If $T \leq 1/2\Omega$ and $f(nT) = 0$ for all $n \in \mathbb{Z}$, then $f \equiv 0$. The rate $\frac{1}{2\Omega}$ is known as the Nyquist rate.

This theorem, though it is simple to state, has far-reaching consequences. The theorem basically states that some signals, or functions, can be perfectly represented by sample values of the function taken at regular, or equispaced, intervals. Knowledge of the frequency bound is required, since this determines the minimum rate at which the signal must be sampled to reconstruct the signal completely. Because Nyquist discovered this minimum rate, it is commonly referred to as the Nyquist rate [8]. No information is lost if the signal is sampled at this rate, and also no additional information is acquired by sampling faster than this rate. That is, the signal is already perfectly reconstructed. Essentially, the band-limited condition guarantees that sufficiently close sampling reveals a nicely behaved function between the sample points. Since f is band-limited, one can gather data sufficiently faster than the signal's oscillation in time and then recover f using the "sinc" function $\frac{\sin(2\pi x)}{x}$ as an interpolator. Thus, one can think of classical sampling as a uniqueness theorem from the theory of analytic splines.

There is also a duality in the theorem. Not only can one reconstruct the function in time, assuming the function is band-limited, but also in frequency, assuming that the function is time-limited. In this case, the function and its transform switch roles.

There is an intriguing history involving the development of this theorem. A dispute prevailed over who initially discovered this result and who is

credited. Consequently, the theorem has several names associated with it: the cardinal series, the Whittaker sampling theorem, the Kotel'nikov theorem, and the Shannon sampling theorem. Often, the names are combined in various forms. For simplicity, it is referred to in this report simply as the classical sampling theorem.

3.2 A Proof

There are numerous proofs of the classical sampling theorem [1,5–7,9,10], some more rigorous and satisfying than others. A simple proof of the theorem follows:

Proof. Let f be Ω band-limited, i.e., $\text{supp}(\widehat{f}) \subset [-\Omega, \Omega]$. Since this condition holds, $\widehat{f}$ can also be represented as the restriction of a 2Ω periodic function to the support of $\widehat{f}$ as

$$\widehat{f}(u) = \sum_{n=-\infty}^{\infty} c_n \exp^{-i\pi n u/\Omega} \cdot \mathcal{X}_{[-\Omega,\Omega]}(u) \ , \tag{9}$$

where the Fourier coefficients are simplified with the use of the Fourier inversion formula and the nature of $\mathcal{X}$. Note, $\mathcal{X}_A$ is the characteristic function defined to be unity in the set A and zero elsewhere. Then the Fourier coefficients are found to be

$$\begin{aligned} c_n &= \frac{1}{2\Omega} \int_{-\Omega}^{\Omega} \widehat{f}(u) \exp^{i\pi n u/\Omega} \, du \\ &= \frac{1}{2\Omega} \int_{-\infty}^{\infty} \widehat{f}(u) \exp^{2i\pi(n/2\Omega)u} \, du \\ &= \frac{1}{2\Omega} f(\frac{n}{2\Omega}) \ . \end{aligned} \tag{10}$$

Substituting equation (10) back into equation (9) and solving for f with the use of the inverse Fourier transform in equation (5) and the uniform convergence of Fourier series, the classical sampling theorem is the result

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} \widehat{f}(u) \exp^{2\pi i u x} \, du \\ &= \int_{-\infty}^{\infty} \frac{1}{2\Omega} \sum_{n=-\infty}^{\infty} f(\frac{n}{2\Omega}) \exp^{-i\pi n u/\Omega} \cdot \mathcal{X}_{[-\Omega,\Omega]} \exp^{2\pi i u x} \, du \\ &= \frac{1}{2\Omega} \sum_{n=-\infty}^{\infty} f(\frac{n}{2\Omega}) \int_{-\Omega}^{\Omega} \exp^{(2\pi i x - \pi i n/\Omega)u} \, du \\ &= \frac{1}{2\Omega} \sum_{n=-\infty}^{\infty} f(\frac{n}{2\Omega}) 2\Omega \frac{\sin \pi(2\Omega x - n)}{\pi(2\Omega x - n)} \\ &= \sum_{n=-\infty}^{\infty} f(\frac{n}{2\Omega}) \frac{\sin \pi(2\Omega x - n)}{\pi(2\Omega x - n)} \ . \end{aligned} \tag{11}$$

The third equality, where the integral and summation are exchanged, is permitted by uniform convergence. So, if $T = 1/2\Omega$, then equation (11) is identical to equation (1).

3.3 An Example

The sampling theory with this band-limited condition, at first, appears too restrictive for practical use. The formula

$$f(t) = T \sum_{n=-\infty}^{\infty} f(nT) \frac{\sin(\frac{\pi}{T}(t - nT))}{\pi(t - nT)}$$

requires that the signal's frequency be bounded for the reconstruction to be stable. However, in practice, this condition is not restrictive. The physical interpretation of an Ω band-limited function is that it is a signal with a frequency no greater than Ω cycles per second. In countless examples, this reflects the reality of the physical world. For instance, many sounds are band-limited naturally; i.e., the frequency of orchestral music is less than 20 kH and the human voice is less than 8 kH. Even our hearing is limited to a certain range of frequencies, and outside of these frequencies, we miss the signal. For instance, not many people can hear the frequency of a dog whistle.

Suppose, one had a signal that was Ω band-limited. Then its Fourier transform would be equivalent to the product of $S(\omega)$ and $F(\omega) = \mathcal{X}_{[-\Omega,\Omega]}(\omega)$, the characteristic function of $[-\Omega, \Omega]$. Then the signal is defined by the convolution of the inverse Fourier transform of $S(\omega)$, perhaps $s(x)$, and $f(x) = \frac{\sin(2\pi x)}{\pi x}$. For simplicity, let $\Omega = 1$ and the function $S(\omega) = 1$, then the signal is represented just by $f(x)$. (See fig. 1.)

According to the sampling theorem, since f is band-limited, it can be reconstructed from its values at the points nT, where $T \leq 1/2\Omega = 1/2$ and $n \in \mathbb{Z}$:

$$g(x) = T \sum_{n=-\infty}^{\infty} f(nT) \frac{\sin(\frac{\pi}{T}(x - nT))}{\pi(x - nT)} \quad ,$$

where $x \in \mathbb{R}$.

In the reconstruction in figure 2, I set $T = 1/2\Omega = 1/2$. Because of the band-limited condition, there is no gain to oversampling. To further illustrate the significance of the band-limited condition, i.e., the need to sample at or above the Nyquist rate, the signal function is undersampled. The next sample, in figure 3, is taken at $T = \frac{1}{2*0.8} \geq \frac{1}{2\Omega}$, i.e., below Nyquist.

Knowledge of the frequency bound allows one to periodically extend the transform to the real line. But when the sample is taken below the Nyquist

Figure 1. Original signal $f(x)$ and its Fourier transform $f(\omega)$.

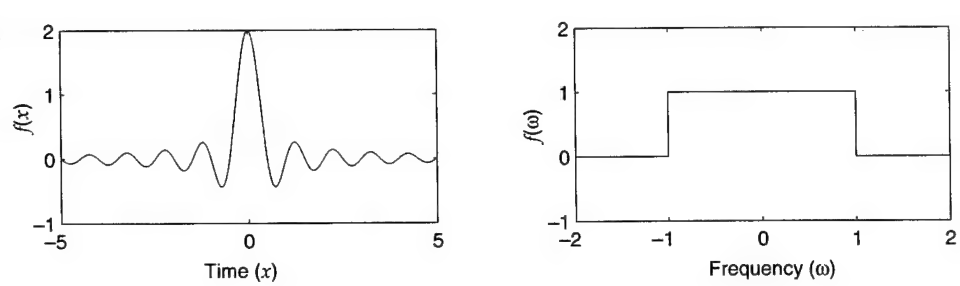


Figure 2. Original signal f and reconstruction g.

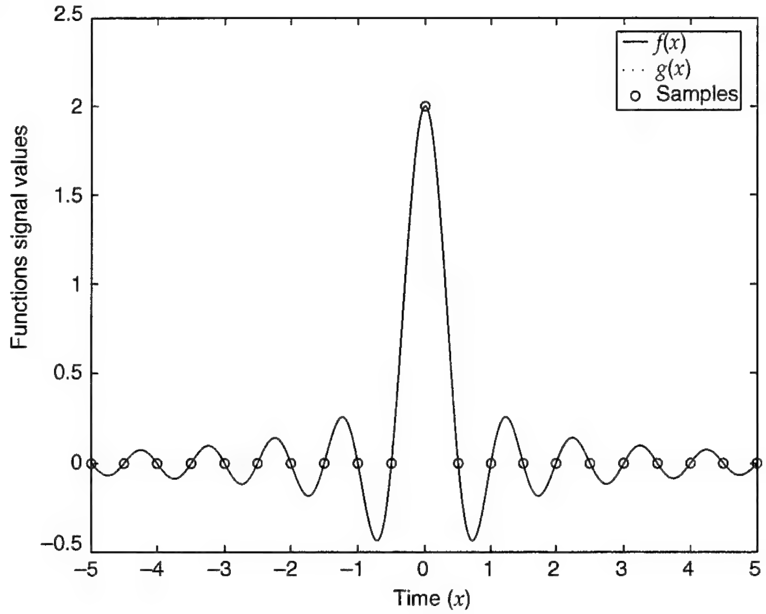
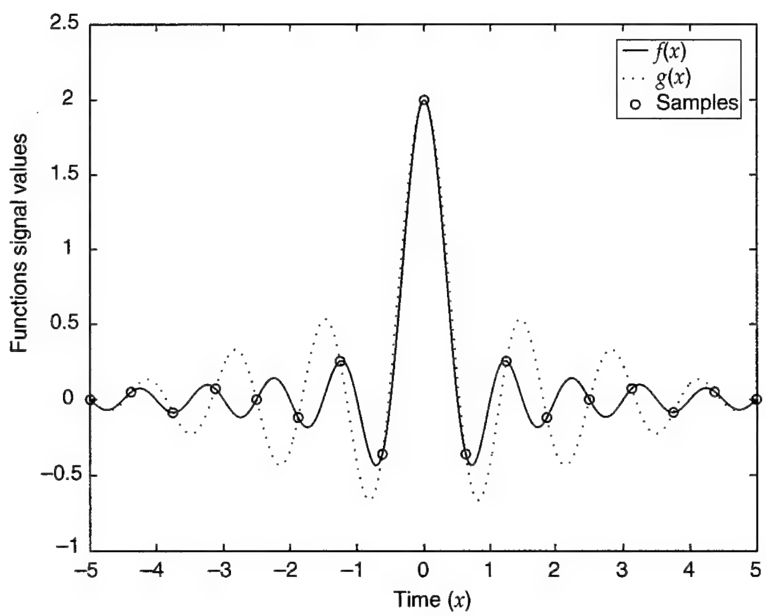


Figure 3. Aliasing caused by undersampling.



rate, those periodic extensions contain overlapping sections and the information is not weighted appropriately. This effect, demonstrated in figure 3, is known as aliasing.

3.4 Deriving the Cardinal Series

There are numerous ways in which to derive the cardinal series, or sampling formula, as attested in the literature concerning this. The historical development of its construction follows in section 3.5. In this report, I display several methods that successfully derive the cardinal series. The first, which is called the delta method [5], is natural in its approach but is not rigorous! The second derivation uses the Cauchy integral formula and shows this to be equivalent to the classical sampling theorem under certain conditions.

3.4.1 Delta Method

For this derivation, the function is sampled equidistantly. Mathematically, this is using a series of generalized functions with point support, i.e., *delta* functions. Thus, a sampled version of f is

$$f_s(t) = f(t) \sum_{n\in\mathbb{Z}} \delta(t - n\tau) = \sum_{n\in\mathbb{Z}} f(n\tau)\delta(t - n\tau) \ , \tag{12}$$

where τ controls the rate of the sampling. This hints at the following reconstruction formula that uses some function g:

$$f(t) = \sum_{n\in\mathbb{Z}} f(n\tau)g(t - n\tau) \ . \tag{13}$$

The task is now to find an appropriate interpolating function g and rate τ. Then equation (13) can be rewritten as

$$f(t) = \int_{\mathbb{R}} f(u)g(t - u) \sum_{n\in\mathbb{Z}} \delta(u - n\tau) \, du \ . \tag{14}$$

The Fourier series expansion of the delta sum is $\sum_{n\in\mathbb{Z}} \delta(u - n\tau) = 1/\tau \sum_{n\in\mathbb{Z}} \exp^{-2\pi inu/\tau}$ with the use of equation (12) [1]. So, again equation (14) is rewritten as

$$\begin{aligned} f(t) &= \int_{\mathbb{R}} f(u)g(t - u)\frac{1}{\tau} \sum_{n\in\mathbb{Z}} \exp^{-2\pi inu/\tau} \, du \\ &= f(\cdot) \sum_{n\in\mathbb{Z}} \exp^{-2\pi in\cdot/\tau} * \frac{1}{\tau} g(\cdot)(t) \ , \end{aligned} \tag{15}$$

where the convolution operator $*$ is defined in section 2. Then, the Fourier transform of both sides results in

$$\widehat{f}(x) = \frac{1}{\tau}\widehat{g}(x) \sum_{n\in\mathbb{Z}} \widehat{f}(x + \frac{n}{\tau}) \ . \tag{16}$$

Thus, the transform of f is equivalent to some weighted spectral repetition of itself. To make sense, g must be such that it chooses only one copy of the spectrum. This is inherently impossible if the repetitions overlap. Supposing that f is ω band-limited, then $1/\tau \geq 2\omega$ is needed to prevent overlapping repetitions. This reveals the importance of the rate $\tau \leq 1/2\omega$ in sampling the function. Now, $\widehat{g}(x)/\tau$ is a window in which to keep the desired copy of the spectrum, where $n = 0$. Thus, define g so that

$$\widehat{g}(x) = \tau \cdot \mathcal{X}_{[-\frac{1}{2\tau},\frac{1}{2\tau}]}(x) \ , \tag{17}$$

and note that $\widehat{g}$ needs only to be equivalent to the characteristic function over the interval $[-\omega,\omega] \subset [-1/2\tau, 1/2\tau]$. This is equivalent to equation (17) and keeps the consistency in the formula dependent on the sampling rate exclusively. This reveals the interpolator to be

$$g(t) = \tau \cdot \frac{\sin 2\pi \frac{1}{2\tau} t}{\pi t} = \frac{\sin \frac{\pi}{\tau} t}{\frac{\pi}{\tau} t} \ , \tag{18}$$

and therefore, the final rewriting of equation (13) becomes the sampling formula

$$\begin{aligned} f(t) &= \sum_{n\in\mathbb{Z}} f(n\tau) g(t - n\tau) \\ &= \sum_{n\in\mathbb{Z}} f(n\tau) \frac{\sin \frac{\pi}{\tau}(t - n\tau)}{\frac{\pi}{\tau}(t - n\tau)} \ . \end{aligned} \tag{19}$$

Again, the classical sampling formula is derived. This is not the only method of deriving the classical sampling theorem. A method that uses complex variables is demonstrated in the next section (3.4.2).

3.4.2 Complex Method

The sampling theorem is intimately connected to a wide variety of mathematical theories. Under certain conditions, it has been shown to be equivalent to a number of other important formulas: the Poisson summation formula, the Cauchy integral formula, etc [5]. Its connection to Cauchy integral theory is demonstrated here.

The relation between complex integration theory and the sampling formula is inherently obvious. The contour integral around a set of simple poles is simply a linear combination of some function's values at the poles (of course, with the poles removed). This is a kind of sampling relationship, made obvious when one considers that in the classical sampling theorem, the "sinc" function contains the appropriate zero values at the sampled data points of the signal function. This technique fits under the umbrella of general complex interpolation. The splitting of the integral is called the Jacobi interpolation formula. This technique was used by Berenstein, Casey, Yger, and Walnut to solve deconvolution problems.

To show this connection mathematically, I will start with Cauchy's integral formula. See Marsden and Hoffman [11] for the necessary background in complex analysis.

Cauchy Integral Formula Theorem [11]. Let f be analytic on a region D and let C be a closed curve in D that is homotopic to a point, i.e., let C be a simple, closed, and rectifiable curve in $\mathbb{C}$ and also let $z \in D$ be a point in D inside C. Then, for $r \in \mathbb{N}_0$,

$$\eta(C, z) \cdot f^{(r)}(z) = \frac{r!}{2\pi i} \oint_C \frac{f(\gamma)}{(\gamma - z)^{r+1}} \, d\gamma \ , \tag{20}$$

where $\eta(C, z) = 1/2\pi i \oint_C 1/(\gamma - z)\, d\gamma$ is the winding number. Note, $\eta(C, z) = 1$ if C is positively oriented and loops around $z \in \operatorname{int} C$ only once.

Given this theorem, I can derive the classical sampling formula. For simplicity, assume that $\Omega \leq 1/2T$ and place the sample values at the integers, $n \in \mathbb{Z}$. Thus, the sampling rate $T = 1$ and $\Omega \leq 1/2$. Let f be a function that satisfies the conditions in the theorem. By the Paley-Weiner theorem in section 2, f is an analytic function in $t \in \mathbb{R}$, which analytically continues to the entire complex plane. Moreover, this entire function satisfies the Paley-Weiner growth bound, i.e., for $\Omega > 0$,

$$|f(z)| \leq A \exp^{-2\pi\Omega|\mathrm{Im} z|} = \frac{A_1}{(1 + |\zeta|)^N} \exp^{-2\pi\Omega|\mathrm{Im} z|} \quad . \tag{21}$$

Now, define $\widehat{\mu}(\zeta) = \sin \pi\zeta$. This has zeros at the integer values, i.e., the sampling rate. So the contour will need to avoid the zeros of $\widehat{\mu}(\zeta)$. I define C_m to be a square contour with corners on

$$(\pm 1 \pm i)(m + \frac{1}{2}) \ .$$

Thus, $\eta(C, z) = 1$. Now, first use the Jacobi interpolation formula (see Casey and Walnut [12]) that gives

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_{C_m} \frac{f(\zeta)}{\zeta - z} \, d\zeta \\ &= \frac{1}{2\pi i} \oint_{C_m} \frac{f(\zeta)}{\zeta - z} \cdot \frac{\widehat{\mu}(\zeta)}{\widehat{\mu}(\zeta)} \, d\zeta \\ &= \frac{1}{2\pi i} \oint_{C_m} \frac{f(\zeta)}{\zeta - z} \cdot \frac{\widehat{\mu}(\zeta) - \widehat{\mu}(z)}{\widehat{\mu}(\zeta)} \, d\zeta + \frac{1}{2\pi i} \oint_{C_m} \frac{f(\zeta)}{\zeta - z} \cdot \frac{\widehat{\mu}(z)}{\widehat{\mu}(\zeta)} \, d\zeta \\ &= \qquad\qquad (1) \qquad\qquad\qquad + \qquad\qquad (2) \ . \end{aligned} \tag{22}$$

Next, I shall show that (2) $\to 0$ as $m \to \infty$, where

$$(2) = \frac{\widehat{\mu}(z)}{2\pi i} \oint_{C_m} \frac{f(\zeta)}{(\zeta - z)\widehat{\mu}(\zeta)} \, d\zeta \ . \tag{23}$$

Recall, that C_m is a square contour with corners $(\pm 1 \pm i)(m + 1/2)$. Let R be a closed, compact disc centered at 0. Then when $m + 1/2 > R$ and $\zeta \in C_m$, $|\zeta - z| \geq m + 1/2 - R$, such that

$$\begin{aligned} |(2)| &= \left|\frac{\widehat{\mu}(z)}{2\pi i} \oint_{C_m} \frac{f(\zeta)}{(\zeta - z)\widehat{\mu}(\zeta)} \, d\zeta\right| \\ &\leq \frac{|\widehat{\mu}(z)|}{2\pi} \oint_{C_m} \frac{|f(\zeta)|}{|(\zeta - z)| \cdot |\widehat{\mu}(\zeta)|} \, d\zeta \\ &\leq \frac{|\widehat{\mu}(z)|}{2\pi(m + \frac{1}{2} - R)} \oint_{C_m} \frac{|f(\zeta)|}{|\widehat{\mu}(\zeta)|} \, d\zeta \ . \end{aligned} \tag{24}$$

Now, let $\zeta = u + iv$. Then, define $C_m^{(1)}$ as the side of C_m that joins $(1 - i)(m+1/2)$ and $(1+i)(m+1/2)$, $C_m^{(2)}$ that joins $(1+i)(m+1/2)$ and $(-1+i)(m+1/2)$, $C_m^{(3)}$ that joins $(-1+i)(m+1/2)$ and $(-1-i)(m+1/2)$, and $C_m^{(4)}$ that completes the square contour that joins $(-1-i)(m+1/2)$ and $(1-i)(m+1/2)$.

On the side $C_m^{(1)}$, $\zeta = m + 1/2 + iv$, thus

$|\sin(\pi\zeta)|^2 = \sin^2(\pi u) + \sinh^2(\pi v) = 1 + \sinh^2(\pi v) = \cosh^2(\pi v) \geq \exp^{2\pi|v|}$.

Also $|f(\zeta)| \leq A \exp^{-2\pi\Omega|v|}$ in equation (21), so that if $\Omega < \frac{1}{2}$,

$$\begin{aligned} \oint_{C_m^{(1)}} \frac{|f(\zeta)|}{|\widehat{\mu}(\zeta)|} d\zeta &\leq A \int_{-m-\frac{1}{2}}^{m+\frac{1}{2}} \exp^{-\pi|v|(2\Omega+1)} dv \\ &= A \frac{\exp^{-\pi(2\Omega+1)(m+\frac{1}{2})} - 1}{\pi(2\Omega+1)} , \end{aligned} \tag{25}$$

which disappears as $m \to \infty$, since $\Omega > 0$. However, if $\Omega = 1/2$, then

$$\begin{aligned} \oint_{C_m^{(1)}} \frac{|f(\zeta)|}{|\widehat{\mu}(\zeta)|} d\zeta &\leq \int_{-m-\frac{1}{2}}^{m+\frac{1}{2}} \frac{A_1}{(1+|\zeta|)^N} d\zeta \\ &= \int_{-m-\frac{1}{2}}^{m+\frac{1}{2}} \frac{A_1}{(1+(m+\frac{1}{2})^2+v^2)^N} dv \\ &= \frac{A_1 \cdot (2m+1)}{(1+(m+\frac{1}{2})^2)^N} . \end{aligned} \tag{26}$$

Thus, the integral goes to 0 as $m \to \infty$. A similar approach can show the same for $C_m^{(3)}$. On the side $C_m^{(2)}$, $\zeta = u + i(m+1/2)$, and $|\sin(\pi\zeta)| \geq \sinh(\pi(m+1/2))$. Thus, if $\Omega \leq 1/2$, then

$$\begin{aligned} \oint_{C_m^{(2)}} \frac{|f(\zeta)|}{|\widehat{\mu}(\zeta)|} d\zeta &\leq \int_{-m-\frac{1}{2}}^{m+\frac{1}{2}} \frac{A_1}{(1+u^2+(m+\frac{1}{2})^2)^N} \frac{\exp^{-2\pi|m+\frac{1}{2}|\Omega}}{\sinh\pi(m+\frac{1}{2})} du \\ &= \frac{A_1 \cdot (2m+1)}{(1+(m+\frac{1}{2})^2)^N} \frac{\exp^{-2\pi\Omega(m+\frac{1}{2})}}{\sinh\pi(m+\frac{1}{2})} , \end{aligned} \tag{27}$$

which also disappears as $m \to \infty$, since $\Omega > 0$. Likewise, $C_m^{(4)}$ is integrated in the same manner. So, equation (22) becomes

$$\begin{aligned} f(z) = (1) &= \frac{1}{2\pi i} \oint_{C_m} \frac{f(\zeta)}{\zeta - z} \cdot \frac{\widehat{\mu}(\zeta) - \widehat{\mu}(z)}{\widehat{\mu}(\zeta)} d\zeta \\ &= \frac{\widehat{\mu}(z)}{2\pi i} \oint_{C_m} \frac{f(\zeta)}{\zeta - z} \cdot \left[\frac{1}{\widehat{\mu}(z)} - \frac{1}{\widehat{\mu}(\zeta)} \right] d\zeta . \end{aligned} \tag{28}$$

At this point, I need to note a couple items before proceeding. First,

$$\sin\pi\zeta = \lim_{N\to\infty} s_N(\zeta) = \lim_{N\to\infty} \pi\zeta \prod_{j=1}^{N} \left(1 + \frac{\zeta^2}{j^2}\right) \tag{29}$$

is the product representation of a sine function. Second, note the following partial fractions decomposition:

$$\frac{1}{(\zeta - z)s_N(\zeta)} = \frac{1}{(\zeta - z)s_N(z)} + \sum_{|n|\le N} \frac{1}{(\zeta - n)(n - z)s_N'(n)} \, . \tag{30}$$

To show this, I need the following: Let $p(\zeta)$ be a polynomial with only simple zeros at $\{r_j\}_{j=1}^{s}$, then (shown in app A)

$$\frac{1}{p(\zeta)} = \sum_{j=1}^{s} \frac{1}{(\zeta - r_j)p'(r_j)} \, . \tag{31}$$

Then, if $m < N$ and defining $R_m = (2)$ on C_m,

$$\begin{aligned}
f(z) &= \frac{\widehat{\mu}(z)}{2\pi i} \oint_{C_m} \frac{f(\zeta)}{\zeta - z} \cdot \left[\frac{1}{\widehat{\mu}(z)} - \frac{1}{\widehat{\mu}(\zeta)}\right] d\zeta + R_m \\
&= \lim_{N\to\infty} \frac{s_N(z)}{2\pi i} \oint_{C_m} \frac{f(\zeta)}{\zeta - z} \cdot \left[\frac{1}{s_N(z)} - \frac{1}{s_N(\zeta)}\right] d\zeta + R_m \\
&= \lim_{N\to\infty} \frac{s_N(z)}{2\pi i} \oint_{C_m} \cdot \sum_{|n|\le N} \frac{f(\zeta)}{(\zeta - n)(n - z)s_N'(n)} \, d\zeta + R_m \\
&= \lim_{N\to\infty} \frac{s_N(z)}{2\pi i} \sum_{|n|\le N} \cdot \oint_{C_m} \frac{f(\zeta)}{(\zeta - n)(n - z)s_N'(n)} \, d\zeta + R_m \qquad (32) \\
&= \lim_{N\to\infty} s_N(z) \sum_{|n|\le m} \frac{f(n)}{(z - n)s_N'(n)} + R_m \\
&= \sin \pi z \sum_{|n|\le m} f(n) \cdot \frac{(-1)^n}{\pi(z - n)} + R_m \\
&= \sum_{|n|\le m} f(n)\mathrm{sinc}(z - n) + R_m \, .
\end{aligned}$$

Thus, as $m \to \infty$, then $R_m \to 0$, and this is the classical sampling theorem. After the limit, the contour integral covers over all the zeros of the sampling function and thus integrates over the whole function. Note, the switch between limit and integration in the second equality and between summation and integration in the fourth is permitted because of uniform convergence. A similar approach is undertaken by Higgins [5], and he also demonstrates other and more general equivalence relations involving the classical sampling formula.

3.5 A Bit of History

Sampling theory grew out of a plethora of other mathematical fields. It is intimately connected with interpolation theory and approximation theory. In fact, for certain cases, it can be shown that the cardinal series (the classical sampling formula), the Cauchy integral formula, and the Poisson summation formula are equivalent [5]. Sampling theory also has applications

in numerous other fields of systems theory, prediction theory, information theory, stochastic processes, signal and image processing, etc. The theory of interpolation grew out of the need to calculate intermediate values of functions with some known values. In 1624, Briggs introduced methods for achieving this by using successive and modified differences. Wallis also had some influence in this area in the 1650s and is credited with coining the term "interpolation." However, it was not until the work of Gregory and Newton that the polynomial nature of these methods began to be recognized. In 1670, Gregory introduced the first interpolation series, and later Newton introduced divided and adjusted divided differences. Both methods derive from taking the polynomial as a close approximation to the (continuous) function to be interpolated [13].

Although these methods were somewhat successful in providing approximations, it became desirable to construct a polynomial interpolant without the need of finding successive differences. This was achieved with the identity below:

$$\begin{aligned} f(\lambda_0) + (t-\lambda_0)f(\lambda_0,\lambda_1) + \cdots + (t-\lambda_0)\ldots(t-\lambda_{M-1})f(\lambda_0,\,\ldots\,,\lambda_M) \\ = \sum_{j=0}^{M} f(\lambda_j)\frac{G_M(t)}{G'_M(\lambda_j)(t-\lambda_j)}\ , \end{aligned} \tag{33}$$

where

$$\begin{aligned} f(\lambda_0,\lambda_1) &= \frac{f(\lambda_1)-f(\lambda_0)}{\lambda_1-\lambda_0} \\ f(\lambda_0,\lambda_1,\lambda_2) &= \frac{f(\lambda_1,\lambda_2)-f(\lambda_0,\lambda_1)}{\lambda_2-\lambda_0} \\ &\ldots \end{aligned}$$

and

$$G_M(t) = \prod_{j=0}^{M}\left(1-\frac{t}{\lambda_j}\right) .$$

Both sides are identical because both are polynomials of degree M taking on the same values on $\{\lambda_j\}$. The right side of the equality in equation (33) is known as Lagrange's formula. Essentially, the Lagrange interpolator acts as a delta function over the known values. This formula first appeared in Lagrange's lectures given in 1795. However, it was previously discovered by Waring in 1779 [5].

Parallel to the above approach, it became desirable to have a periodic interpolant rather that a polynomial one. Given a periodic function of the form $p(t) = \sum_{j=-M}^{M} c_j \exp^{ijt}$, then appropriate formulas were found for these cases. One such formula is

$$p(t) = \frac{1}{2M+1}\sum_{j=0}^{2M} p\left(\frac{2\pi j}{2M+1}\right)\frac{\sin[(\frac{2M+1}{2})(t-\frac{2\pi j}{2M+1})]}{\sin\frac{1}{2}(t-\frac{2\pi j}{2M+1})} .$$

This finite sampling series is credited to Cauchy (1841), although Gauss introduced primitive forms as far back as 1805 [5]. Cauchy's other significant contribution was to be the first to note the importance of the rate of the sampling.

The research conducted in the nineteenth century extended Lagrange's formula from interpolation on a finite number of points to the more general case of infinitely many points $\{\lambda_n\}$. Assuming $f(\lambda_n) = a_n$ for all $n \in \mathbb{N}$,

$$f(z) = \sum_n a_n \frac{\phi(z)}{\phi'(\lambda_n)(z - \lambda_n)} ,$$

where ϕ is a function with simple zeros at $\{\lambda_n\}$. A more general formula was also constructed:

$$\sum_n a_n \left(\frac{z}{\lambda_n}\right)^{s_n} \frac{\phi(z)}{\phi'(\lambda_n)(z - \lambda_n)} ,$$

where (s_n) is a sequence of integers appropriately chosen to ensure convergence [14]. These formulas were found useful by Cazzaniga in 1882 for constructive purposes. The century ended with the first explicit statement of the cardinal series by Borel. In 1899, Borel set $\lambda_n = n$ and noted that $\sin(\pi z)$ has the appropriate zeros, returning the classical sampling formula [5].

During the first half of the twentieth century, this special case of the Lagrange formula with knowledge of equidistantly spaced points, i.e., the cardinal series, was rediscovered in this form several times [5,14]. Whipple did so in 1910, and E. T. Whittaker in 1915 [10]. Whittaker would also point out the band-limited nature of the sum. His son, J. M. Whittaker, would coin the phrase "cardinal series" in 1929. Ogura, in 1920, was the first to provide a fully rigorous proof of the cardinal series by using the calculus of residues. In 1928, Nyquist established the time-bandwidth component of a signal, the importance of rates, in connection with telegraphy. His work showed that stability and reconstruction were not possible below a certain sampling rate. In 1941, Hardy noticed that the cardinal series is an orthogonal expansion, an important development contributing to Hilbert and Banach spaces. In 1933, Kotel'nikov introduced sampling theory into communication theory. However, this fact was unknown in the west until the 1960s. For the Western world, Shannon introduced it in 1949 [14]. Shannon's original statement of the theorem is "If a function $f(t)$ contains no frequencies higher than W cps, it is completely determined by giving its ordinates at a series of points spaced (1/2 W) s apart."

In fact, he claims that this fact is common knowledge in communication and credits Whittaker for an earlier form of the theorem. He also notes Nyquist's and Gabor's use of this fact. However, his name remains attached to the theorem.

Since Shannon's use of the sampling theorem, much has been developed in this field of research. Parzen [15] in 1956 and Petersen and Middleton [16] in 1962 extended the sampling theorem to multiple dimensions, and Kramer developed a generalization of the classical sampling theorem in 1959 [17].

4. Multirate Sampling Theory

The classical sampling theorem shows that one can effectively recover a signal using a regular sampling rate determined by knowledge of the signal frequency's bandwidth. However, if the sampling rate is slower than the Nyquist rate, the function cannot be reconstructed via the classical theorem. Research in this direction has proven effective via other methods. The procedure here relies on both the classical theorem and deconvolution theory to show how a band-limited function that is sampled below Nyquist can still be completely recovered.

4.1 Multichannel Deconvolution

Deconvolution has proven itself to be a useful mathematical tool in the field of signal and image processing. For images, it acts as an enhancing filter to correct blurs in a picture. For signals, it can be used to correct distorted line shapes without a loss of signal-noise ratio. In essence, deconvolution is useful in signal and image processing, where the incoming data contains a high degree of information. The reason this works so naturally is because the convolution equation models a number of linear systems.

Before I progress further, it is appropriate to understand what convolution is defined as mathematically. Recall the definition of convolution, also given earlier (sect. 2), for $f, g \in L^1(\mathbb{R})$, the convolution of f and g is defined by

$$f * g(x) = \int f(y)g(x-y)\,dy$$

for $x \in \mathbb{R}$.

For the purpose of this report, the convolution equation $s = f * \mu$ models linear, translation invariant systems (e.g., sensors, linear filters). In this model, f is the input signal function, μ is the system impulse response distribution, and s is the output (received) signal. However, in many physical applications, s is often a poor approximation of the signal f. This motivates one to deconvolve f from μ to attain the original signal. Results have shown that if the convolver μ is time-limited (i.e., compactly supported) and nonsingular (i.e., not a delta generalized function), then this problem is *ill-posed*, in the sense of Hadamard [12]. It has been shown to be ill-posed for all realizable convolvers—all convolvers that can be built. For the circumvention of this scenario, a theory of multichannel deconvolution has been developed to solve these equations. A multichannel system preserves information about the signal that would otherwise be lost. Thus, data lost

by one convolver can still be retained by another convolver. The signal is now overdetermined, i.e.,

$$s_i = f * \mu_i, i = 1 \dots n \ .$$

Then, if the convolvers $\{\mu_i\}$ satisfy the condition of being strongly coprime, then deconvolving for f is now a well-posed problem.

Strongly coprime. Strongly coprime is defined as a set of convolvers $\{\mu_i\}$ that satisfy the inequality

$$\left(\sum_{i=1}^{n} \widehat{\mu}_i|(\zeta)|^2\right)^{\frac{1}{2}} \geq A \exp^{-B|\mathrm{Im}z|} \cdot (1+|\zeta|)^{-N} \tag{34}$$

for every $\zeta \in \mathbb{C}$, where A and B are positive constants and N is a positive integer, is said to be strongly coprime.

Note, this tells us that $1/\sum_{i=1}^{n} |\widehat{\mu}_i(\zeta)|^2$ satisfies the Paley-Weiner growth bound. If this is so, then there exists a set of time-limited deconvolvers $\{\nu_i\}$ such that

$$\mu_1 * \nu_1 + \cdots + \mu_n * \nu_n = \delta \ ,$$

and consequently,

$$\widehat{\mu}_1 \cdot \widehat{\nu}_1 + \cdots + \widehat{\mu}_n \cdot \widehat{\nu}_n = 1 \ ,$$

where δ is the *Dirac delta* function. The second of these two equations is the analytic Bezout equation. The existence of such deconvolvers is guaranteed by the following theorem:

Hörmander [12]. For compactly supported distributions $\{\mu_i\}_{i=1}^{n}$ on $\mathbb{R}$, compactly supported distributions $\{\nu_i\}_{i=1}^{n}$ exist such that

$$\delta = \mu_1 * \nu_1 + \cdots + \mu_n * \nu_n \ .$$

That is, they satisfy the analytic Bezout equation if and only if the set of distributions $\{\mu_i\}_{i=1}^{n}$ is strongly coprime.

By Hörmander's theorem, a strongly coprime set has a solution in the analytic Bezout equation. Thus, given the deconvolvers and output signals, f is naturally produced:

$$\sum_i s_i * \nu_i = \sum_i (f * \mu_i) * \nu_i = \sum_i f * (\mu_i * \nu_i) = f * \sum_i (\mu_i * \nu_i) = f * \delta = f \ .$$

The system is such that no information is lost in this process. This occurs because the condition of being strongly coprime guarantees that the zeros in the analytic Bezout equation do not cluster quickly as $|\zeta| \to \infty$. If the $\{\widehat{\mu}_i\}$ did have a common zero, then $\widehat{s}_i(\zeta) = 0$ at that zero and information about f would be lost. Because the system is engineered toward eliminating any common zero, no information about f is lost (and the problem is

well-posed). Thus, the signal f is gathered by this strongly coprime system, and these received signals are then filtered by the deconvolvers to reconstruct f.

These methods are linear and realizable; thus deconvolution at a time sample only depends on the information near that time sample. Unfortunately, Hörmander's theorem is an existence theorem; it does not reveal what the deconvolvers might be. One obvious solution is

$$\nu_i = \frac{\overline{\mu_i}}{\sum_i |\mu_i|^2} \ .$$

The deconvolvers have been shown not to be unique, and so, for certain scenarios, one set of deconvolvers may be better than another. More in this direction is given by Casey and Walnut [18].

4.2 Noncommensurate Sampling Lattices

The theory developed here merges together the two ideas of multichannel deconvolution and classical sampling theory. It has been developed by Casey [19], Casey and Sadler [20], and Casey and Walnut [12,18].

By the results in multichannel deconvolution, a set of strongly coprime convolvers $\{\mu_i\}$ needs to be created for the problem to be well-posed. First, a definition is needed, then the following theorem will be useful in creating this set.

Definition. A real number α is said to be poorly approximated by rationals if an integer $N \geq 2$ and a constant $C = C(\alpha)$ exist such that for all integers p, q with $q \geq 2$,

$$\left| \alpha - \left(\frac{p}{q}\right) \right| \geq C|q|^{-N} \ . \tag{35}$$

This class of numbers will be denoted by $\mathbb{P}$.

Theorem. Let $0 < r_1 < \cdots < r_m$, $m \geq d = 1$ satisfy the condition that for all $i \neq j$, r_i/r_j are in $\mathbb{P}$, then $\{\mathcal{X}_{[-r_i, r_i]^d}\}$ is a strongly coprime set. Then to create a set of appropriate convolvers, one needs only to find such a set whose ratios are inadequately approximated by rationals.

4.2.1 Two Sampling Lattices

Suppose $f \in L^2(\mathbb{R})$. Let α be an irrational in the class $\mathbb{P}$. Now, the pair $\mu_1 = \mathcal{X}_{[-1,1]}, \mu_2 = \mathcal{X}_{[-\alpha,\alpha]}$ are strongly coprime. Thus, the problem of solving the analytic Bezout equation for ν_1 and ν_2

$$\widehat{\mu}_1 \cdot \widehat{\nu}_1 + \widehat{\mu}_2 \cdot \widehat{\nu}_2 = 1$$

is well-posed. Now, $\widehat{\mu}_1(\omega) = \frac{\sin(2\pi\omega)}{\pi\omega}$ and $\widehat{\mu}_2(\omega) = \frac{\sin(2\pi\alpha\omega)}{\pi\omega}$ have zeros

$$Z_{\widehat{\mu}_1} = \left\{\frac{n}{2}\right\} \ , \ Z_{\widehat{\mu}_2} = \left\{\frac{n}{2\alpha}\right\}$$

for $n \in \mathbb{Z}\backslash\{0\}$. Thus, one solution to the analytic Bezout equation is $\widehat{\nu}_1(\omega) = 1/2\widehat{\mu}_1(\omega)$ for $\omega \in Z_{\widehat{\mu}_1}$ and, likewise, $\widehat{\nu}_2(\omega) = 1/2\widehat{\mu}_2(\omega)$ for $\omega \in Z_{\widehat{\mu}_2}$. Then, the problem simply becomes an interpolation problem on the set of zeros $\Gamma = Z_{\widehat{\mu}_1} \bigcup Z_{\widehat{\mu}_2}$. So, one has a new reconstruction formula for f from its values on Γ. However, this can be done for any poorly approximated rational. The following theorem results:

Theorem. Let

$$\Gamma = \left\{\frac{\pm k}{2}\right\} \bigcup \left\{\frac{\pm k}{2\alpha}\right\} \tag{36}$$

for $n \in \mathbb{N}$, and let $\lambda_i \in \Gamma$. Let f be a $(1+\alpha)$-band-limited function, then f can be conditionally reconstructed by the formula

$$f(t) = \sum_{\lambda_i \in \Gamma} f(\lambda_i) \frac{G(t)}{G'(\lambda_i)(t-\lambda_i)} + [f(0)K_1(t) + f'(0)K_2(t)] \ , \tag{37}$$

where

$$G(t) = \sin(2\pi t) \cdot \sin(2\pi\alpha t) \tag{38}$$

$$K_1(t) = \frac{G(t)}{\frac{G''(0)}{2!} t^2} \tag{39}$$

$$K_2(t) = \frac{G(t)}{\frac{G''(0)}{2!} t} \ . \tag{40}$$

Note that the interpolators at the origin appropriately model the function at the origin, i.e., $K_1(0) = 1$ while $K_1'(0) = 0$ and $K_2(0) = 0$ while $K_2'(0) = 1$. How these interpolators are calculated will be demonstrated later in section 4.3.1. Also, note that the information in the signal can be reconstructed uniquely by its samples on $\Gamma \cup \{0\}$. Here, it is also important to note that the sampling rates correspond to 1-band-limited functions and α-band-limited functions. These add up to the band-limit of the function to be sampled. Thus, a $(1+\alpha)$-band-limited function is reconstructed via significantly lower sampling rates.

4.2.2 An Arbitrary Number of Lattices

This result generalizes for an arbitrary number of sampling rates. Let $\{r_i\}_{i=1}^n$ be rates such that all r_i/r_j, $i \neq j$, are in $\mathbb{P}$, the class of irrationals poorly approximated by rationals. Then, $\{r_i\}$ is a strongly coprime set. Thus, our convolvers have the form $\mu_i(t) = \mathcal{X}_{[-r_i,r_i]}(t)$. These $\{\mu_i\}$ model the impulse response of a multichannel system. The convolvers have Fourier transforms

$$\widehat{\mu}_i(\zeta) = \frac{\sin(2\pi r_i \zeta)}{\pi\zeta}$$

with zero sets $Z_i = \{\frac{\pm k}{2r_i}\}$, both respectively, for $k \in \mathbb{N}$. Note that exclusive of the origin, the zeros sets are nonrepetitive. Let $\Gamma_i = Z_i$ and $\Gamma = \bigcup_{i=1}^n \Gamma_i =$

$\{\lambda_i\}$. Thus, f can be reconstructed on its values at Γ. This can also be done via techniques in complex interpolation theory.

Thus, the following theorem results: Let $\{r_i\}_{i=1}^{n}$ be a set whose ratios are poorly approximated by rationals, and let f be a $(\sum r_i)$-band-limited function. Let

$$\Gamma_i = \left\{ \frac{\pm k}{2r_i} \right\}$$

for $k \in \mathbb{N}$ and $i = 1, \ldots, n$. Also, let

$$\bigcup_{i=1}^{n} \Gamma_i = \Gamma = \{\lambda_i\} \quad .$$

Then f is uniquely determined by

$$\{f(\lambda_i)\} \quad \bigcup \quad \left\{ f(0), \ldots, f^{(n-1)}(0) \right\} \quad .$$

Furthermore, f can be reconstructed from its values on $\Gamma \bigcup \{0\}$ by the following formula:

$$f(t) \approx \sum_{\lambda_i \in \Gamma} f(\lambda_i) \frac{G(t)}{G'(\lambda_i)(t - \lambda_i)} + \sum_{i=1}^{n} f^{(i-1)}(0) \cdot K_i(0) \quad , \qquad (41)$$

where

$$G(t) = \prod_{i=1}^{n} \sin(2\pi r_i t) \qquad (42)$$

and the interpolating functions $K_i(t)$ at the origin are a linear combination of $G(t)/t^j$, $j = 1, \ldots, n$, chosen so that $K_i^{(l-1)}(0) = \delta_{i,l}$, $i, l = 1, \ldots, n$.

4.3 Interpolators at Origin

What remains for this interpolating function in equation (41) is to find appropriate interpolators at the origin. To remove this ambiguity, I construct interpolators here. Under the assumption that there are two interpolators, then $G(z) = \sin(2\pi z) \cdot \sin(2\pi \alpha z)$, where α is the square root of any prime number. What is desired, as stated in the theorem in section 4.2.2, is for the interpolators to satisfy the following condition: $\{K_i(t)\}$ is a linear combination of $G(t)/t^i$, $i = 1, \ldots, n$, chosen so that $K_i^{(j-1)}(0) = \delta_{i,j}$, where $j = 1, \ldots, n$.

4.3.1 Two Interpolators

Assume that there are two interpolators. As stated earlier in equation (42),

$$G(t) = \sin(2\pi t) \cdot \sin(2\pi \alpha t) \ .$$

To find the appropriate interpolators at the origin, I need to manipulate this function. Remember that the Taylor expansion formula of a sine function is

$$\sin(t) = t - \frac{t^3}{3!} + O(t^5) \ . \tag{43}$$

Thus, inserting equation (43) into $G(t)$ obtains

$$\begin{aligned} G(t) &= \sin(2\pi t) \cdot \sin(2\pi\alpha t) \\ &= \left[2\pi t - \frac{(2\pi t)^3}{3!} + O(t^5)\right] \cdot \left[2\pi\alpha t - \frac{(2\pi\alpha t)^3}{3!} + O(t^5)\right] \\ &= 4\pi^2\alpha t^2 - \left[\frac{16\pi^4\alpha^3}{3!} + \frac{16\pi^4\alpha}{3!}\right] t^4 + O(t^6) \ . \end{aligned}$$

And so, the following are representations for the linear combinations of $G(t)/t^i$, $i = 1, \dots, n$:

$$\begin{aligned} \frac{G(t)}{t} &= 4\pi^2\alpha t - \left[\frac{16\pi^4\alpha^3}{3!} + \frac{16\pi^4\alpha}{3!}\right] t^3 + O(t^5) \\ \frac{G(t)}{t^2} &= 4\pi^2\alpha - \left[\frac{16\pi^4\alpha^3}{3!} + \frac{16\pi^4\alpha}{3!}\right] t^2 + O(t^4) \ . \end{aligned}$$

Their respective derivatives follow:

$$\begin{aligned} \frac{d}{dt}\frac{G(t)}{t} &= 4\pi^2\alpha - 3\left[\frac{16\pi^4\alpha^3}{3!} + \frac{16\pi^4\alpha}{3!}\right] t^2 + O(t^4) \\ \frac{d}{dt}\frac{G(t)}{t^2} &= 2\left[\frac{16\pi^4\alpha^3}{3!} + \frac{16\pi^4\alpha}{3!}\right] t + O(t^3) \ . \end{aligned}$$

Next, note the limits of the functions and their derivatives at $t = 0$ —

$$\begin{aligned} \lim_{t\to 0} \frac{G(t)}{t} &= 0 \\ \lim_{t\to 0} \frac{G(t)}{t^2} &= 4\pi^2\alpha \\ \lim_{t\to 0} \frac{d}{dt}\frac{G(t)}{t} &= 4\pi^2\alpha \\ \lim_{t\to 0} \frac{d}{dt}\frac{G(t)}{t^2} &= 0 \ . \end{aligned}$$

The first interpolant K_1 is desired to be such that $K_1(0) = 1$ and $K_1'(0) = 0$. Also, K_2 is desired such that $K_2(0) = 0$ and $K_2'(0) = 1$. Setting the following relations satisfy these requirements:

$$K_1(t) = \frac{G(t)}{\frac{G''(0)}{2!} t^2} \ , \quad K_2(t) = \frac{G(t)}{\frac{G''(0)}{2!} t} \ , \tag{44}$$

where, conveniently, one can see that $G''(0)/2! = 4\pi^2\alpha$.

4.3.2 Arbitrary Number of Interpolators

It is convenient to further construct this idea to an arbitrary multiple of interpolators. In this scenario, assume that there are n interpolators involved. Thus, as the theorem in section 4.2.2 dictates, there are n sampling rates $\{r_i\} = \{\sqrt{p_0} = 1, \sqrt{p_1}, \ldots, \sqrt{p_{n-1}}\}$, where $p_1, \ldots, p_{n-1}$ are the first $n-1$ primes, which generate the following reconstruction interpolant:

$$G(t) = \prod_{i=1}^{n} \sin(2\pi r_i t) \ . \tag{45}$$

Now, the process of finding the interpolants at the origin continues as before. With the use of the Taylor expansion formula for the sine function, then

$$\begin{aligned}
G(t) &= \prod_{i=1}^{n} \left[2\pi r_i t - \frac{8\pi^3 r_i^3}{3!} t^3 + O(t^5) \right] \\
&= \prod_{i=1}^{n} \left[2\pi r_i t - \frac{8\pi^3 p_{i-1} \cdot r_i}{3!} t^3 + O(t^5) \right] \\
&= 2^n \pi^n \left(\prod_{i=1}^{n} r_i \right) t^n - \frac{2^{n+2}\pi^{n+2}}{3!} \left(\prod_{i=1}^{n} r_i \right) \left(1 + \sum_{i=1}^{n-1} p_i \right) t^{n+2} + O(t^{n+4}) \\
&= 2^n \pi^n \sqrt{p_1 p_2 \cdots p_n}\, t^n - \frac{2^{n+2}\pi^{n+2}}{3!} \sqrt{p_1 p_2 \cdots p_n} \left(1 + \sum_{i=1}^{n-1} p_i \right) t^{n+2} + O(t^{n+4}) \ .
\end{aligned} \tag{46}$$

Now, the interpolants at the origin $\{K_i\}$ again will be linear combinations of $G(t)/t^i, i = 1, \ldots, n$, chosen appropriately so that $K_i^{(j-1)}(0) = \delta_{i,j}$,where $j = 1, \ldots, n$. The general form of $G(t)/t^i$ is

$$\frac{G(t)}{t^k} = 2^n \pi^n \sqrt{p_1 p_2 \cdots p_n}\, t^{n-k} - \frac{2^{n+2}\pi^{n+2}}{3!} \sqrt{p_1 p_2 \cdots p_n} \left(1 + \sum_{i=1}^{n-1} p_i \right) t^{n-k+2} + O(t^{n-k+4}) \ . \tag{47}$$

Let $H_k(t) = G(t)/t^k$. Then, the formula for the j^{th} derivative of H_k is

$$\begin{aligned}
H_k^{(j)}(t) &= 2^n \pi^n \frac{(n-k)!}{(n-k-j)!} \sqrt{p_1 p_2 \cdots p_n}\, t^{n-k-j} \\
&\quad - \frac{2^{n+2}\pi^{n+2}}{3!} \frac{(n-k+2)!}{(n-k-j+2)!} \sqrt{p_1 p_2 \cdots p_n} \left(1 + \sum_{i=1}^{n-1} p_i \right) t^{n-k-j+2} + O(t^{n-k-j+4}) \ .
\end{aligned} \tag{48}$$

Allowing t to approach 0, the following results are obtained:

$$\begin{aligned}
\lim_{t \to 0} H_k(t) &= 2^n \pi^n \sqrt{p_1 p_2 \cdots p_n}\, \delta_{n,k} \\
&= \frac{G^{(n)}(0)}{n!} \ ,
\end{aligned} \tag{49}$$

and

$$
\begin{aligned}
\lim_{t\to 0} H_k^{(j)}(t) &= 2^n\pi^n \frac{(n-k)!}{(n-k-j)!}\sqrt{p_1p_2\cdots p_n}\,\delta_{n,k+j} \\
&\quad -\frac{2^{n+2}\pi^{n+2}}{3!}\frac{(n-k+2)!}{(n-k-j+2)!}\sqrt{p_1p_2\cdots p_n}\left(1+\sum_{i=1}^{n-1} p_i\right)\delta_{n,k+j-2} \ .
\end{aligned} \tag{50}
$$

Now, using these evaluations in equation (50) as before, I can construct a formula of appropriate interpolants using the set $\{H_k\}$. First, note that each interpolant is of the form

$$
K_k(t) = \sum_{i=1}^{k} c_i H_i(t)
$$

for all $k = 1, \ldots, n$. Note that the c_i's vary with the k's. As required of the sampling theorem, coefficients c_i are needed such that $K_k^{(j)}(0) = \delta_{k-1,j}$. By linearity of differentiation, I have

$$
\begin{aligned}
K_k^{(j)}(0) &= \sum_{i=1}^{k} c_i H_i^{(j)}(0) \\
&= \sum_{i=1}^{k} c_i \cdot \left[2^n\pi^n \frac{(n-i)!}{(n-i-j)!}\sqrt{p_1p_2\cdots p_n}\,\delta_{n,i+j} \right. \\
&\quad \left. -\frac{2^{n+2}\pi^{n+2}}{3!}\frac{(n-i+2)!}{(n-i-j+2)!}\sqrt{p_1p_2\cdots p_n}\left(1+\sum_{i=1}^{n-1} p_i\right)\delta_{n,i+j-2}\right] \\
&= \sum_{i=1}^{k} \left[c_i 2^n\pi^n \frac{(n-i)!}{(n-i-j)!}\sqrt{p_1p_2\cdots p_n} \right. \\
&\quad \left. -c_{i+2}\frac{2^{n+2}\pi^{n+2}}{3!}\frac{(n-i)!}{(n-i-j)!}\sqrt{p_1p_2\cdots p_n}\left(1+\sum_{i=1}^{n-1} p_i'\right)\right]\delta_{n,i+j} \qquad (51) \\
&= \sum_{i=1}^{k} 2^n\pi^n \frac{(n-i)!}{(n-i-j)!}\sqrt{p_1p_2\cdots p_n}\left[c_i - c_{i+2}\frac{2^2\pi^2}{3!}\left(1+\sum_{i=1}^{n-1} p_i\right)\right]\delta_{n,i+j} \\
&= 2^n\pi^n\sqrt{p_1p_2\cdots p_n}\cdot j!\left[c_{n-j} - c_{n-j+2}\frac{2^2\pi^2}{3!}\left(1+\sum_{i=1}^{n-1} p_i\right)\right] \\
&= \frac{G^{(n)}(0)}{(n)!}\cdot j!\left[c_{n-j} - c_{n-j+2}\frac{2^2\pi^2}{3!}\left(1+\sum_{i=1}^{n-1} p_i\right)\right] \\
&= \delta_{k-1,j} \ .
\end{aligned}
$$

In equation (51), $c_{k+1}, c_{k+2} = 0$ is defined so the summation evaluates correctly. Now, remember the following relation must hold: $K_k^{(k-1)}(0) = 1$,

since

$$K_k^{(k-1)}(0) = \frac{G^{(n)}(0)}{(n)!} \cdot (k-1)! \left[c_{n-k+1} - c_{n-k+3} \frac{2^2\pi^2}{3!} \left(1 + \sum_{i=1}^{n-1} p_i \right) \right] . \quad (52)$$

Since $K_k^{(j)}(0) = 0$ if $j \neq k-1$, then

$$\left[c_{n-j} - c_{n-j+2} \frac{2^2\pi^2}{3!} \left(1 + \sum_{i=1}^{n-1} p_i \right) \right] = 0 . \quad (53)$$

Thus, via cancelation, $c_{n-k+1} = (G^{(n)}(0)/(n)! \cdot (k-1)!)^{-1}$. Note, also from the summations just mentioned, that there is the restriction $n - j \leq k$, or restated as $j \geq n - k$. So, continuing in this direction, I find the following recursive equations:

$$c_{n-k+1+2m} = \frac{[\frac{2^2\pi^2}{3!}(1 + \sum_{i=1}^{n-1} p_i)]^m}{\frac{G^{(n)}(0)}{(n)!} \cdot (k-1)!} \quad (54)$$

$$c_{n-k+2+2m} = 0 . \quad (55)$$

This generates interpolants at the origin with the desired properties. Thus, the interpolants can be formulated as

$$\begin{aligned} K_{n-k+1}(t) &= \sum_{i=1}^{n-k} c_i H_i(t) \\ &= \frac{\sum_{m=0}^{\lfloor (n-k)/2 \rfloor} \left(\frac{2^2\pi^2}{3!} \left(1 + \sum_{i=1}^{n-1} p_i \right) \right)^m t^{2m} G(t)}{\frac{G^{(n)}(0)}{(n)!} \cdot (k-1)! \cdot t^{n-k+1}} . \end{aligned} \quad (56)$$

Now, with these interpolants, I can use this new reconstruction method to recover a function of a larger bandwidth than our individual sampling rates.

4.4 Simulations

Here I present a number of simulations of the reconstructions of a signal using the multirate sampling formula. The same function (signal) used in section 3.3, $f(t) = \sin(2\pi t)/\pi t$. Recall that this is an $\Omega = 1$ band-limited function. The classical sampling theory stipulates that the sampling rate be $T \leq 1/2\Omega = 1/2$. The classic theory dictates that the Nyquist rate condition holds to successfully reconstruct the function. As shown earlier, it was sufficient for the signal to be sampled at Nyquist. Now, I wish to sample slower than the Nyquist rate and still reconstruct the function. First, the signal will be sampled twice below Nyquist at the sampling values

$$\Gamma_1 = \left\{ \frac{\pm k}{2\frac{1}{(1+\sqrt{2})}} \right\} \quad , \quad \Gamma_2 = \left\{ \frac{\pm k}{2\frac{\sqrt{2}}{(1+\sqrt{2})}} \right\} .$$

Note that these sample rates are both slower than the Nyquist rate. Also, note that these are appropriately chosen, i.e., $1/(1+\sqrt{2})+\sqrt{2}/(1+\sqrt{2}) = 1$, and the ratio of the two samples is $\sqrt{2} \in \mathbb{P}$. Recall, that $\mathbb{P}$ is the class of irrationals poorly approximated by rationals. Thus, by either theorem in section 4.2.2 or 4.2.3, the signal can be reconstructed by

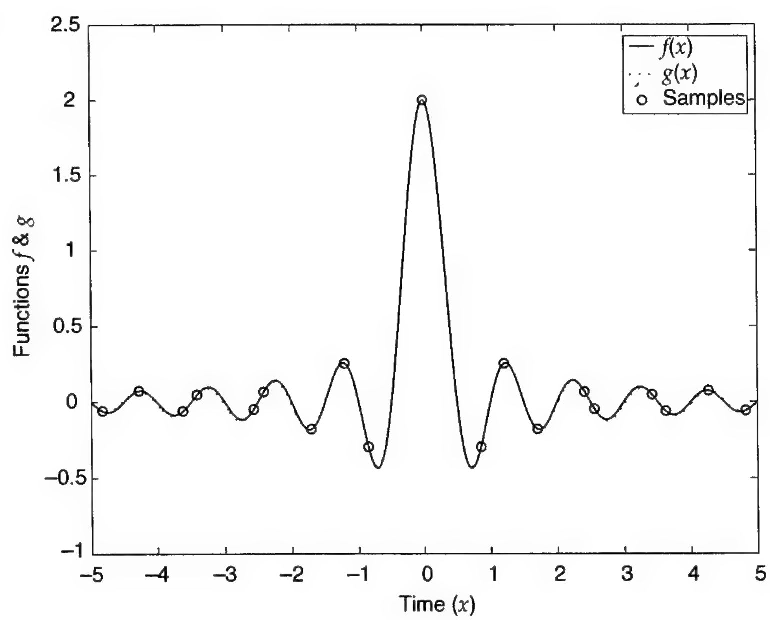
$$g(t) = \sum_{\lambda_i \in \Gamma} f(\lambda_i) \frac{G(t)}{G'(\lambda_i)(t-\lambda_i)} + [f(0)K_1(t) + f'(0)K_2(t)] \ ,$$

with

$$\begin{aligned} G(t) &= \sin\left(\frac{2\pi t}{(1+\sqrt{2})}\right) \cdot \sin\left(\frac{2\pi\sqrt{2}t}{(1+\sqrt{2})}\right) \\ K_1(t) &= \frac{G(t)}{\frac{G''(0)}{2!}t^2} \\ K_2(t) &= \frac{G(t)}{\frac{G''(0)}{2!}t} \ . \end{aligned}$$

Essentially, $G(t)$ looks exactly as it should, i.e,. the product of two sine functions. The reconstruction of the signal is in figure (4). It gives the appearance of completely reconstructing the function signal; however, upon closer analysis, minor variations can be seen between the orignal signal and the reconstruction. These points of aliasing are due to locations in the sampling grid where the samples are *close*. Methods of eliminating this error are discussed by Casey and Walnut [12,19]. The best way to improve this sampling scheme is to back off the bandwidth slightly, i.e., increase the sampling rates by about 10 percent or more. Then, the signal can be reconstructed, practically, within an acceptable margin of error.

Figure 4. Original signal $f(t)$ and bisample reconstruction $g(t)$.



Again, as in the classical formula, the rate is critically important in achieving an accurate reconstruction of the signal. Research on the bounds of these rates is ongoing.

Two items of interest are the interpolator and its derivative and, most importantly, the reconstruction of the signal. The interpolator and its derivative are displayed in figure 5.

In figure 6, I demonstrate the reconstruction of the same signal but use three noncommensurate sampling rates below Nyquist. Here, I use the following sampling rates:

$$\Gamma_1 = \left\{ \frac{\pm k}{2\frac{1}{(1+\sqrt{2}+\sqrt{3})}} \right\} \;,\; \Gamma_2 = \left\{ \frac{\pm k}{2\frac{\sqrt{2}}{(1+\sqrt{2}+\sqrt{3})}} \right\} \;,\; \Gamma_3 = \left\{ \frac{\pm k}{2\frac{\sqrt{3}}{(1+\sqrt{2}+\sqrt{3})}} \right\} .$$

Figure 5. (a) Interpolator and (b) its derivative.

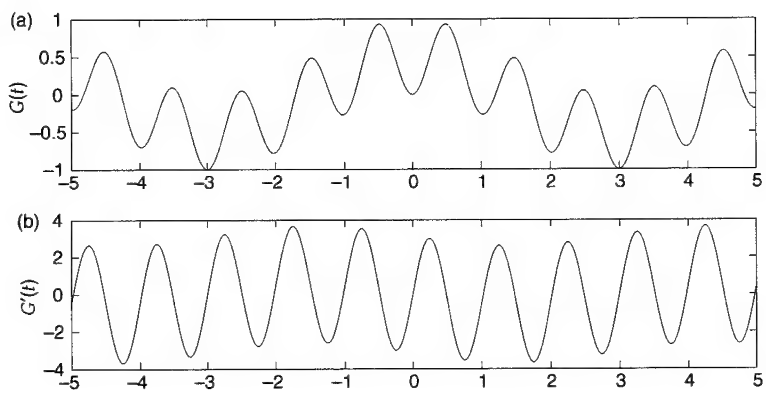
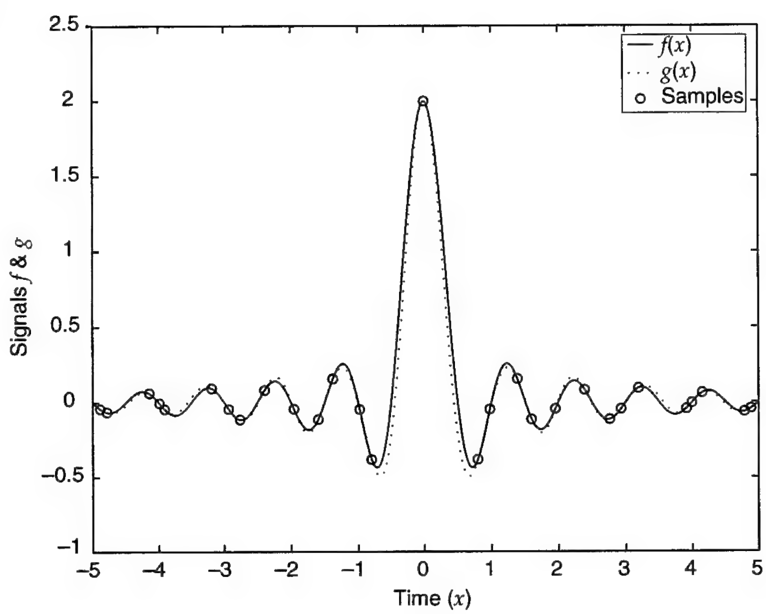


Figure 6. Original signal $f(t)$ and trisample reconstruction $g(t)$.



Note that these rates are still below Nyquist. The reconstruction formula used is the same as presented in section 4.2.2. The only real point of interest to present is the interpolators at the origin. From the derivations in section 4.3.2, the set of interpolators is shown to be

$$\begin{aligned} K_1(t) &= \frac{G(t) + \frac{4\pi^2}{6} \cdot (1 + \sqrt{2} + \sqrt{3}) \cdot t^2 \cdot G(t)}{\frac{G^{(3)}(0)}{3!} t^3} \\ K_2(t) &= \frac{G(t)}{\frac{G^{(3)}(0)}{3!} t^2} \\ K_3(t) &= \frac{G(t)}{\frac{G^{(3)}(0)}{3!} t} \; . \end{aligned}$$

A number of other simulations were run with an encouraging success rate, demonstrating the viability of this multisampling technique. The success rate is somewhat improved when the sampling rates are more equally spaced. A brief discussion of applications follows in the next section.

4.5 Applications

A benefit to using this technique is that the individual sampling rates are lower than the traditional Nyquist sampling rate in the classical sampling theorem. For example, for a $(1+\alpha)$-band-limited function, the Nyquist rate is $1/2(1+\alpha)$, but when sampling with two lattices, the sampling rates are only $1/2$ and $1/2\alpha$. Both these results are below Nyquist. The generalization shows that the comparisons between the sampling rates and the frequency band limit are more outstanding as the number of lattices increase.

Suppose that I have an Ω band-limited signal, but the sampling rate is restricted to $1/2\beta$, then choose the first n primes so that

$$(1 + p_1 + p_2 + \cdots + p_n) \cdot \frac{\beta}{p_n} \geq \Omega \; . \tag{57}$$

Then, $n+1$ sampling lattices can be constructed via the theorem in section 4.2.3 that can effectively reconstruct f with sampling rates no faster than $1/2\beta$.

Unfortunately, this structure has some difficulties. The first is that the sampling grid is extremely rigid; perturbations on the lattices result in losing information. Also, because the sampling points might become arbitrarily close, ripples occur at points near each other. If the bandwidth is backed off, the simulations results improve, but exact bounds are unknown.

These results also extend to higher dimensions, where one can think of the grids and the interpolating function as Cartesian products. Casey and Sadler [20] have used these techniques to develop new analog-digital transformers for signal processing. In their signal processing literature, these are known as A-D converters.

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Appendix A. Proof of the Polynomial Expansion

I shall show the result in section 3.4.2 of the main report via induction on the degree of the polynomial.

Proof. Suppose $p(\zeta)$ is a one degree polynomial with leading coefficient 1 and one simple zero, then $p(\zeta) = \zeta - r_1$. Thus, clearly,

$$\frac{1}{p(\zeta)} = \frac{1}{\zeta - r_1} \; . \tag{A-1}$$

Now, assuming that this statement is true for any k degree polynomial with leading coefficient 1 and k distinct simple zeros, where $k = 1, \ldots, s$, I want to show the same result for an $s + 1$ degree polynomial with s simple zeros. Suppose $p(\zeta)$ is such a polynomial. Then, I have a representation for p using the set of its zeros:

$$p(\zeta) = \prod_{j=0}^{s} (\zeta - r_j) \; . \tag{A-2}$$

I can equivalently state this as $p(\zeta) = (\zeta - r_0) \cdot q(z)$, where q is an s degree polynomial with only s simple zeros. Thus, I have a formula by the assumption in the induction for q. Using this, I obtain the following:

$$\begin{aligned}
\frac{1}{p(\zeta)} &= \frac{1}{(\zeta - r_0)} \cdot \frac{1}{q(z)} \\
&= \frac{1}{(\zeta - r_0)} \cdot \sum_{j=1}^{s} \frac{1}{(\zeta - r_j) \cdot q'(r_j)} \\
&= \sum_{j=1}^{s} \frac{1}{(\zeta - r_0)(\zeta - r_j) q'(r_j)} \; .
\end{aligned} \tag{A-3}$$

Now, let $m(\zeta) = (\zeta - r_0)(\zeta - r_j)q'(r_j)$, then $m'(\zeta) = (\zeta - r_j)q'(r_j) + (\zeta - r_0)q'(r_j)$. Also, note that $p'(\zeta) = q(\zeta) + (\zeta - r_0)q'(\zeta)$. So continuing, I have

$$\begin{aligned}
\frac{1}{p(\zeta)} &= \sum_{j=1}^{s} \frac{1}{(\zeta - r_0)(\zeta - r_j)q'(r_j)} \\
&= \sum_{j=1}^{s} \frac{1}{(\zeta - r_0)m'(r_0)} + \frac{1}{(\zeta - r_j)m'(r_j)} \\
&= \sum_{j=1}^{s} \frac{1}{(\zeta - r_0)(r_0 - r_j)q'(r_j)} + \frac{1}{(\zeta - r_j)(r_j - r_0)q'(r_j)} \qquad \text{(A-4)} \\
&= \frac{1}{(\zeta - r_0)} \sum_{j=1}^{s} \frac{1}{(r_0 - r_j)q'(r_j)} + \sum_{j=1}^{s} \frac{1}{(\zeta - r_j)p'(r_j)} \\
&= \frac{1}{(\zeta - r_0)} \cdot \frac{1}{q(r_0)} + \sum_{j=1}^{s} \frac{1}{(\zeta - r_j)p'(r_j)} \\
&= \frac{1}{(\zeta - r_0)} \cdot \frac{1}{p'(r_0)} + \sum_{j=1}^{s} \frac{1}{(\zeta - r_j)p'(r_j)} .
\end{aligned}$$

Equation A-4 gives the desired result.

Appendix B. MATLAB Code

The following MATLAB *.m files were used to perform the simulations. I used MATLAB version 5.3 by The Mathworks, Inc.

B-1 Shannon.m

```
%SHANNON EXAMPLE % This m-file generates plots for the demonstration of
Shannon's formula. Three plots will be created. The first will show the signal and its
Fourier transform, the second will demonstrate the reconstruction procedure, and
the last plot will show the effects of aliasing in the sampling.

% Terrence Moore, 3/6/2000
% Revision: 1.3   Date: 7/5/2000 18:59:33


%begin of code
%generate the domain
x = linspace(-5,5,1000);
%the signal function
 y = sin(2*pi*x)./(pi*x);

%plot of signal
figure;
subplot(3,2,3);
plot(x,y);
title('original signal f(x)');
xlabel('x');
ylabel('f(x)');

%the transform of the signal
x1 = [-2 -1 -1 1 1 2];
y1 = [0 0 1 1 0 0];

%plot of the transform
subplot(3,2,4);
plot(x1,y1);
axis([-2 2 -1 2]);
title('Fourier transform f(\ omega)');
xlabel('\ omega');
ylabel('f(\ omega)');
```

```
%our recontruction formula only for this signal
%omega = 1
s=1;
z=0;
count=0;
for n=-20:20
    if n=0
        z=z+sin(pi*n/s)./(pi*n/(2*s)).*sin(pi*(2*s*x-n))./(pi*(2*s*x-n));
        count=count+1;
    end
end
z=z+2*sin(pi*(2*s*x))./(pi*(2*s*x));
figure;
plot(x,y,'-',x,z,':');
hold;
%the sampled data
N = -20:20;
N = N/(2*s);
yN = sin(2*pi*N)./(pi*N);
plot(N,yN,'o');plot(0,2,'o');
axis([-5 5 -0.5 2.5])
xlabel('x');
legend('f(x)','g(x)','samples');

%omega = 0.8 (undersampling - aliasing)
s=0.8;
z=0;
count=0;
for n=-20:20
    if n =0
        z=z+sin(pi*n/s)./(pi*n/(2*s)).*sin(pi*(2*s*x-n))./(pi*(2*s*x-n));
        count=count+1;
    end
end
z=z+2*sin(pi*(2*s*x))./(pi*(2*s*x));
figure;
plot(x,y,'-',x,z,':');
hold;
%the sampled data
N = -20:20;
N = N/(2*s);
yN = sin(2*pi*N)./(pi*N);
```

```
plot(N,yN,'o');plot(0,2,'o');
axis([-5 5 -1 2.5])
xlabel('x');
legend('f(x)','g(x)','samples');

%omega = 1.2 (oversampling -- aliasing)
s=1.2;
z=0;
count=0;
for n=-20:20
    if n =0
        z=z+sin(pi*n/s)./(pi*n/(2*s)).*sin(pi*(2*s*x-n))./(pi*(2*s*x-n));
        count=count+1;
    end
end
z=z+2*sin(pi*(2*s*x))./(pi*(2*s*x));
figure;
plot(x,y,'-',x,z,':');
hold;
%the sampled data
N = -20:20;
N = N/(2*s);
yN = sin(2*pi*N)./(pi*N);
plot(N,yN,'o');plot(0,2,'o');
axis([-5 5 -0.5 2.5])
title('reconstruction function g(x) and signal f(x)');
xlabel('x');
```

B-2 MultirateShannon.m

```
t = linspace(-5,5,1000);

y = sin(2*pi*t)./(pi*t);

r0 = 1 + sqrt(2);
r = [1 sqrt(2)];
r = r./r0;
N = 20;
rates = 1./(2*r);
Ns = round(N*(1./rates) +1);
tsamp = zeros(length(r),2*max(Ns)+1);
for i = 1:length(r)
tsamp(i,:) = [(-1*Ns(i):Ns(i)) zeros(1,2*max(Ns)-2*Ns(i))];
end
samp = diag(rates)*tsamp;
samp = samp(:);
indz = find(samp == 0);
samp(indz) = [];
lambda = sort([samp.']);
lambda = lambda;% + .001;
ST = sin(2*pi*r.'*t);
St = prod(ST,1);
CT = 2*pi*diag(r)*cos(2*pi*r.'*t);
CT = flipud(CT);
Sp = sum(ST.*CT,1);

figure;
subplot(2,1,1);
plot(t,St);
subplot(2,1,2);
plot(t,Sp);

Sl = sin(2*pi*r.'*lambda);
Cl = 2*pi*diag(r)*cos(2*pi*r.'*lambda);
Cl = flipud(Cl);
bp = Sl.*Cl;
bp = sum(bp,1);
%sample values to be interpolated
tp = sin(2*pi*lambda)./(pi*lambda);
fp = tp./bp;
```

```
temp = zeros(size(t));
for i = 1:length(lambda)
temp = temp + fp(i)./(t-lambda(i));
end

f = temp.*St;

ST2 = 2*4*pi*pi*prod(r,2);
K1 = St./(t.*t);
K1 = K1/ST2*2;
K2 = St./(t);
K2 = K2/ST2*2;

f = f + 2*K1 + 0*K2;

figure;
plot(t,y,'-',t,f,':');
hold;
%the sampled data
N = lambda;
yN = sin(2*pi*N)./(pi*N);
plot(N,yN,'o');plot(0,2,'o');
axis([-5 5 -1 2.5])
xlabel('x');
legend('f(x)','g(x)','samples');
```

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Sampling theory is that branch of mathematics that seeks to reconstruct functions from its values at a discrete set of points. The fundamental result in sampling theory known as "Shannon's sampling theorem" has many applications to signal processing and communications engineering. I demonstrate Shannon's result via complex interpolation methods. I then quote a result that uses these methods to solve interpolation problems on unions of noncommensurate lattices, which are created via a specific number of theoretic guidelines. These interpolations give Shannon-type reconstructions on these lattices. I close by doing simulations in MATLAB of the sampling reconstructions on these noncommensurate grids.

14. SUBJECT TERMS Signal processing, Shannon's Sampling Theorem, multichannel deconvolution sampling lattices, interpolation, Cauchy Integral Formula, signal reconstruction, multirate sampling

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